ON THE STRATIFICATION BY X-RANKS OF A LINEARLY NORMAL ELLIPTIC CURVE $X \subset \mathbb{P}^n$

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Abstract. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve. For any $P \in \mathbb{P}^n$ the X-rank of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. Here we give an almost complete description of the stratification of $\mathbb{P}^n$ given by the X-rank.

Fix an integral and non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a subset $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span. The X-rank is an extensively studied topic ([8], [5], [4] and references therein). In the applications one needs only the cases in which X is either a Veronese embedding of a projective space or a Segre embedding of a multiprojective space. We feel that the general case gives a treasure of new projective geometry. Up to now only for rational normal curves there is a complete description of the stratification of $\mathbb{P}^n$ by X-rank ([7], [8], Theorem 5.1, [4]). Here we look at the case of elliptic linearly normal curves. For any integer $t \geq 1$ let $\sigma_t(Y)$ denote the closure in $\mathbb{P}^n$ of all $(t-1)$-dimensional linear spaces spanned by t points of Y. Set $\sigma_0(Y) = \emptyset$. For any $P \in \mathbb{P}^n$ the border X-rank $b_X(P)$ is the minimal integer $t \geq 1$ such that $P \in \sigma_t(X)$, i.e. the only positive integer t such that $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$. If (as always in this paper) Y is a curve, then $\dim(\sigma_t(Y)) = \min\{n, 2t-1\}$ for all $t \geq 1$ ([1], Remark 1.6). Notice that $r_X(P) \geq b_X(P)$ and that equality holds at least on a non-empty open subset of $\sigma_t(X) \setminus \sigma_{t-1}(X)$, $t := b_X(P)$. Obviously $b_X(P) = 1 \iff P \in X \iff r_X(P)$. Thus to compute all X-ranks it is sufficient to compute the X-ranks of all points of $\mathbb{P}^n \setminus X$. In this paper we compute it for the linearly normal elliptic curve. We prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n \setminus X$ and set $w := b_X(P)$. We have $2 \leq w \leq \lfloor (n+2)/2 \rfloor$. Assume $n \geq 2w$. Then either $r_X(P) = w$ or $r_X(P) = n+1-w$ and both cases occurs for some $P \in \sigma_w(X) \setminus \sigma_{w-1}(X)$.*

The inequalities $2 \leq w \leq \lfloor (n+2)/2 \rfloor$ in the statement of Theorem 1 are obvious ([1], Remark 1.6). The case $w = 2$ and arbitrary n was settled in [4], Theorem 3.13. Theorem 1 leaves partially open the cases $n = 2w-1$ and $n = 2w-2$. If $n = 2w-1$, then we may have $r_X(P) = w$ and $r_X(P) \geq w+1$ (see Propositions 3 and 2), but we are not able to rule out the case $r_X(P) = w+2$. If $n = 2w-2$ we are in the dark. The case $n = 3$ is contained in [9] (here we have $r_X(P) \leq 3$ and

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in characteristic zero to get this inequality it is sufficient to quote [8], Proposition 4.1).

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. This assumption is essential in our proofs, mainly to quote [6], Proposition 5.8, which is a very strong non-linear version of Bertini's theorem.

1. Preliminary lemmas

In this paper an elliptic curve is a smooth and connected projective curve with genus 1.

The following lemma and its proof is just a reformulation of [2], Lemma 1.

Lemma 1. *Let $Y \subset \mathbb{P}^r$ be an integral variety. Fix any $P \in \mathbb{P}^r$ and two zero-dimensional subschemes A, B of Y such that $A \neq B$, $P \in \langle A\rangle$, $P \in \langle B\rangle$, $P \notin \langle A'\rangle$ for any $A' \subsetneqq A$ and $P \notin \langle B'\rangle$ for any $B' \subsetneqq B$. Then $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(1)) > 0$.*

Proof. Since A and B are zero-dimensional, we have the inequality $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(1)) \geq \max\{h^1(\mathbb{P}^n, \mathcal{I}_A(1)), h^1(\mathbb{P}^r, \mathcal{I}_B(1))\}$. Thus we may assume $h^1(\mathbb{P}^r, \mathcal{I}_A(1)) = h^1(\mathbb{P}^r, \mathcal{I}_B(1)) = 0$, i.e. $\dim(\langle A\rangle) = \deg(A)-1$ and $\dim(\langle B\rangle) = \deg(B)-1$. Set $D := A\cap B$ (scheme-theoretic intersection). Thus $\deg(A\cup B) = \deg(A)+\deg(B)-\deg(D)$. Since $D \subseteq A$ and A is linearly independent, we have $\dim(\langle D\rangle) = \deg(D) - 1$. Since $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\dim(\langle A \cup B\rangle) \leq \deg(A \cup B) - 1$, we get $h^1(\mathbb{P}^r, \mathcal{I}_{A\cup B}(1)) > 0$ if and only if $\langle D\rangle \subsetneqq \langle A\rangle \cap \langle B\rangle$ Since $A \neq B$, $D \subsetneqq A$. Hence $P \notin \langle D\rangle$. Since $P \in \langle A\rangle \cap \langle B\rangle$, we are done. □

Notation 1. Let $C \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Let $\beta(C)$ be the maximal integer such that every zero-dimensional subscheme of C with degree at most $\beta(C)$ is linearly independent.

Proposition 1. *Fix an integer $k \leq \lfloor \beta(C)/2\rfloor$ and any $P \in \sigma_k(C)\setminus\sigma_{k-1}(C)$. Then there exists a unique zero-dimensional scheme $Z \subset C$ such that $\deg(Z) \leq k$ and $P \in \langle Z\rangle$. Moreover $\deg(Z) = k$ and $P \notin \langle Z'\rangle$ for all $Z' \subsetneqq Z$.*

Proof. The existence part is stated in [3], Lemma 1, which in turn is just an adaptation of some parts of the beautiful paper [5] ([5], Lemma 2.1.6) or of [4], Proposition 11. The uniqueness part is true by Lemma 1 and the definition of the integer $\beta(C)$. □

Remark 1. Let $X \subset \mathbb{P}^n$ be a linearly normal elliptic curve.

(i) Since X is projectively normal, the cohomology of line bundles on X gives $\beta(X) = n$ and that a zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = n+1$ is not linearly independent if and only if $Z \in |\mathcal{O}_X(1)|$.

(ii) Fix zero-dimensional schemes $A, B \subset X$ such that $\deg(A)+\deg(B) = n+1$. If $\mathcal{O}_X(A+B) \neq \mathcal{O}_X(1)$, then the degree $n+1$ divisor $A+B$ (different from $A\cup B$ if $A\cap B \neq \emptyset$) is linearly independent and $\langle A\rangle \cap \langle B\rangle = \langle A\cap B\rangle$ (scheme-theoretic intersection). Hence B does not evince $r_X(P)$ for any $P \in \langle A\rangle \cap \langle B\rangle$, unless $B\cap A = B$, i.e. $B \subseteq A$. If $\mathcal{O}_X(A+B) \cong \mathcal{O}_X(1)$, then $\dim(\langle A\rangle\cap\langle B\rangle) = \deg(A\cap B)$.

(iii) Fix zero-dimensional schemes $A, B \subset X$ such that $\deg(A) + \deg(B) \leq n$, and $\langle A\rangle \cap \langle B\rangle \neq \emptyset$. Fix any $P \in \langle A\rangle \cap \langle B\rangle$. Since $A \cup B$ is linearly dependent, Lemma 1 implies that at least one among the schemes A and B, say A, has a proper subscheme A' such that $P \in \langle A'\rangle$. Take as A' a minimal such subscheme. Thus $P \notin A''$ for any $A'' \subsetneqq A'$. Apply the same trick to A' and B. We get $A' \subseteq B$. At the end we get $\langle A\rangle \cap \langle B\rangle = \langle A \cap B\rangle$. Thus if B evinces $r_X(P)$, then $B \subseteq A$.

Fix any non-degenerate variety $X \subset \mathbb{P}^n$. For any $P \in \mathbb{P}^n$ let $\mathcal{S}(X,P)$ denote the set of all $S \subset X$ evincing $r_X(P)$, i.e. the set of all $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that every $S \in \mathcal{S}(X,P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$. Now assume that X is a linearly normal elliptic curve. Let $\mathcal{Z}(X,P)$ denote the set of all zero-dimensional subschemes $Z \subset X$ such that $\deg(Z) = b_X(P)$ and $P \in \langle Z \rangle$. Lemma 3 below gives $\mathcal{Z}(X,P) \neq \emptyset$. Fix any $Z \in \mathcal{Z}(X,P)$. Notice that Z is linearly independent (i.e. $\dim(\langle Z \rangle) = \deg(Z) - 1$) and $P \notin \langle Z' \rangle$ for any subscheme $Z' \subsetneqq Z$.

Lemma 2. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix $P \in \mathbb{P}^n$. Then either $b_X(P) = r_X(P)$ or $r_X(P) + b_X(P) \geq n + 1 - b_X(P)$.*

Proof. Assume $b_X(P) < r_X(P)$. Fix W evincing $b_X(P)$ and S evincing $r_X(P)$. Assume $\sharp(S) + \deg(W) \leq n$. Thus $S \cup W$ is linearly independent (Remark 1), i.e. $\langle S \rangle \cap \langle W \rangle = \langle W \cap S \rangle$. Since S is reduced, while W is not reduced, $W \cap S \subsetneqq W$. Thus $b_X(P) \leq \deg(W \cap S) < b_X(P)$, a contradiction. □

Lemma 3. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a linearly normal elliptic curve. Fix a positive integer w such that $2w \leq n + 1$. Fix $P \in \mathbb{P}^n$ and assume the existence of a zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = w$, $P \in \langle Z \rangle$, while $P \notin \langle Z' \rangle$ for all $Z' \subsetneqq Z$. Then $b_X(P) = w$.*

Proof. Assume $b_X(P) < w$ and take a scheme $B \in \mathcal{Z}(X,P)$ (Proposition 1). Hence $P \in \langle B \rangle$ and $\deg(B) \leq w - 1$. Since $\deg(Z) + \deg(B) \leq n$, $Z \cup B$ is linearly independent. Thus $\langle Z \rangle \cap \langle B \rangle = \langle Z \cap B \rangle$. We have $P \in \langle Z \rangle \cap \langle B \rangle$. Since $\deg(B) < w$, we have $Z \cap B \subsetneqq Z$. Hence $P \notin \langle Z \cap B \rangle$, a contradiction. The converse part follows from Proposition 1, part (i) of Remark 3 and the inequality $2w \leq n + 1$. The last assertion follows from the first part using induction on the integer $b_X(Q)$. □

2. Proofs and related results

Proposition 2. *Fix an integer $k \geq 1$, a linearly normal elliptic curve $C \subset \mathbb{P}^{2k+1}$ and $P \in \mathbb{P}^{2k+1} \setminus \sigma_k(C)$.*

(a) Either $\sharp(\mathcal{Z}(C,P)) \leq 2$ or $\mathcal{Z}(C,P)$ is infinite. We have $Z_1 \cap Z_2 = \emptyset$ and $\mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1)$ for any $Z_1, Z_2 \in \mathcal{Z}(C,P)$ such that $Z_1 \neq Z_2$.

(b) If $\sharp(\mathcal{Z}(C,P)) \neq 2$, then $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$ for all $Z \in \mathcal{Z}(C,P)$.

(c) If $\mathcal{Z}(C,P)$ is infinite, then its positive-dimensional part Γ is irreducible and one-dimensional. Fix a general $Z \in \Gamma$. Either Z is reduced or there is an integer $m \geq 2$ such that $Z = mS_1$ for a reduced $S_1 \subset C$ such that $\sharp(S_1) = (k+1)/m$.

(d) For general P we have $\sharp(\mathcal{Z}(C,P)) = 2$.

Proof. Since no non-degenerate curve is defective ([1], Remark 1.6), we have $\sigma_{k+1}(C) = \mathbb{P}^{2k+1}$ and $\dim(\sigma_k(C)) = 2k - 1$. Thus $b_C(P) = k + 1$. Proposition 1 and part (i) of Remark 1 give $\mathcal{Z}(C,P) \neq \emptyset$. Fix $Z_1, Z_2 \in \mathcal{Z}(C,P)$ such that $Z_1 \neq Z_2$. Part (ii) of Remark 1 gives $\mathcal{O}_C(Z_1 + Z_2) \cong \mathcal{O}_C(1)$ and $Z_1 \cap Z_2 = \emptyset$, proving part (a).

(i) Let $J(C, \dots, C) \subset C^{k+1} \times \mathbb{P}^{2k+1}$ be the abstract join of $k+1$ copies of C, i.e. the closure in $C^{k+1} \times \mathbb{P}^{2k+1}$ of the set of all $(P_1, \dots, P_{k+1}, P)$ such that $P_i \neq P_j$ for all $i \neq j$, $P_1, \dots, P_{k+1}$ is linearly independent and $P \in \langle \{P_1, \dots, P_{k+1}\} \rangle$. Since $\sigma_{k+1}(C) = \mathbb{P}^{2k+1}$, for general P the set $\mathcal{Z}(C,P)$ is finite and its cardinality is the degree of the generically finite surjection $J(C, \dots, C) \to \mathbb{P}^{2k+1}$ induced by the projection $C^{k+1} \times \mathbb{P}^{2k+1} \to \mathbb{P}^{2k+1}$. Assume the existence of schemes $Z_1, Z_2, Z_3 \in \mathcal{Z}(C,P)$ such that $Z_i \neq Z_j$ for all $i \neq j$. Part (a) gives $Z_i \cap Z_j = \emptyset$ and $\mathcal{O}_C(Z_i +$

$Z_j) \cong \mathcal{O}_C(1)$ for all $i \neq j$. Taking $i = 1$ and $j \in \{2,3\}$ we get $\mathcal{O}_C(Z_2) \cong \mathcal{O}_C(Z_3)$. By symmetry we get $\mathcal{O}_C(Z) \cong \mathcal{O}_C(Z_1)$ for all $Z \in \mathcal{Z}(C,P)$. Since $\mathcal{O}_C(Z_1+Z_2) \cong \mathcal{O}_C(1)$, we also get $\mathcal{O}_C(2Z) \cong \mathcal{O}_C(1)$for all $Z \in \mathcal{Z}(C,P)$.

(ii) Now assume $\sharp(\mathcal{Z}(C,P)) = 1$, say $\mathcal{Z}(X,P) = \{Z\}$. Fix any $E \in |\mathcal{O}_C(1)(-Z)|$. Since $E+Z$ is contained in a hyperplane, we have $\langle Z\rangle \cap \langle E\rangle \neq \emptyset$. Part (ii) of Remark 1 gives $\dim(\langle Z\rangle \cap \langle E\rangle) = \deg(Z\cap E)$. Set $J := \{(Q,E) \in \langle Z\rangle \times |\mathcal{O}_C(1)(-Z)| : Q \in \langle E\rangle\}$. We just saw that J is a complete projective set. For dimensional reasons the projection of $\langle Z\rangle \times |\mathcal{O}_C(1)(-Z)|$ into its first factor induces a dominant morphism $u : J \to \langle Z\rangle$. Since J is complete, there is $E \in |\mathcal{O}_C(1)(-Z)|$ such that $u(E) = Z$. The uniqueness of Z gives $E = Z$. Thus $2Z \in |\mathcal{O}_C(1)|$. Since the set of all $Z \subset X$ such that $2Z \in |\mathcal{O}_C(1)|$ has dimension $k+1$, we get $\sharp(\mathcal{Z}(C,P)) = 2$ for a general P, proving part (d). Since this integer is the degree of a generically finite surjection $\gamma : J(C,\dots,C) \to \mathbb{P}^{2k+1}$ and $\mathbb{P}^{2k+1}$ is a normal variety, either $\sharp(\mathcal{Z}(C,P)) \le 2$ or $\mathcal{Z}(C,P)$ is infinite.

(iii) Now assume that $\mathcal{Z}(C,P)$ is infinite. Since any two different elements of $\mathcal{Z}(C,P)$ are disjoint (see step (i)), for a general $A \in C$ there is at most one element of Γ containing A. Thus $\dim(\Gamma) = 1$ and Γ is irreducible. Since a general point of C is contained in a unique element of Γ, the algebraic family Γ of effective divisors of C is a so-called *involution* ([6], §5). Since any two elements of Γ are disjoint, this involution has no base points. Let Z be a general element of Γ. Either Z is reduced or there is an integer $m \ge 2$ such that $Z = mS$ with S reduced ([6], Proposition 5.8), concluding the proof of part (c). □

Proof of Theorem 1. For any integer $k > 0$ such that $\sigma_{k-1}(X) \neq \mathbb{P}^n$, we have $r_X(Q) = k$ for a general $Q \in \sigma_k(X)$. Thus for arbitrary $w \le \lfloor (n+2)/2\rfloor$ there are points P such that $r_X(P) = b_X(P) = w$. Fix $w \le n/2$, P and W such that $b_X(P) = w$, and $r_X(P) > w$. Lemma 2 gives $r_X(P) \ge n+1-w$. Hence to prove Theorem 1 it is sufficient to prove $r_X(P) = n+1-w$. Fix $W \in \mathcal{Z}(X,P)$. Set $\mathcal{B} := \{Z+W\}_{Z\in|\mathcal{O}_X(1)(-2W)|}$. Thus $\mathcal{B} := \{B \in |\mathcal{O}_X(1)(-W)| : W \subset B\}$. Set $\mathcal{S} := \{Z \in |\mathcal{O}_X(1)(-W)| : P \in \langle Z\rangle\}$. Since $\deg(\mathcal{O}_X(1)(-W)) = n+1-w \le n$, every element of $|\mathcal{O}_X(1)(-W)|$ is linearly independent. However, in the definition of the set $\mathcal{S}$ we did not prescribed that $P \notin \langle Z'\rangle$ for all $Z' \subsetneqq Z$. Thus $\mathcal{B} \subseteq \mathcal{S}$. Part (i) of of Remark 1 and the inequality $r_X(P) \ge n+1-w$ give that $r_X(P) = n+1-w$ if and only if there is a reduced $S \in \mathcal{S}$.

(a) Here we prove that $\mathcal{B} \neq \mathcal{S}$. Fix a general subset $E \subset X$ such that $\sharp(E) = n-2w-1$. Since $n > 2w+1$, we have $E \neq \emptyset$. Thus for a general E the degree w line bundles $\mathcal{O}_X(W)$ and $\mathcal{O}_X(1)(-W-E)$ are not isomorphic. Thus to get $\mathcal{B} \neq \mathcal{S}$ it is sufficient to prove the existence of a degree w zero-dimensional subscheme A_E of X such that $E + A_E \in \mathcal{S}$. Let $\ell_{\langle E\rangle} : \mathbb{P}^n \setminus \langle E\rangle \to \mathbb{P}^{2w+1}$ denote the linear projection from $\langle E\rangle$. Call $X_E \subset \mathbb{P}^{2w+1}$ the closure of $\ell_{\langle E\rangle}|(X \setminus \langle E\rangle \cap X)$ in $\mathbb{P}^{2w+1}$. Since X is non-degenerate, X_E spans $\mathbb{P}^{2w+1}$. Since X is a smooth curve, the rational map $\ell_{\langle E\rangle}|(X\setminus\langle E\rangle\cap X)$ extends to a surjective morphism $\psi : X \to X_E$. Since every degree $n-2w+1$ zero-dimensional subscheme of X is linearly normal, E is the scheme-theoretic intersection of X with $\langle E\rangle$. Thus $\deg(X_E)\cdot\deg(\psi) = deg(X) - \deg(E) = n+1-n+2w+1 = 2w+2$. Hence $\deg(X_E) = 2w+2$ and $\deg(\psi) = 1$. Since $\deg(\psi) = 1$, X_E and X are birational. Thus X_E is a linearly normal elliptic curve. Since X and X_E are smooth curves, ψ is an isomorphism. Since $\langle E\rangle \cap X = E$ (as schemes), we have $\psi^*(\mathcal{O}_{X_E}(1)) \cong \mathcal{O}_X(1)(-E)$. Set $W' := \psi(W)$. For general E we

may assume $E \cap W = \emptyset$. Thus W' is a degree w subscheme of X_E isomorphic as an abstract scheme to W. Hence W' is not reduced. Fix $W_1 \subsetneq W'$ and call W_2 the only subscheme of W such that $\psi(W_2) = W_1$. Since W' is linearly independent, $\ell_{\langle E\rangle}|\langle W\rangle \to \langle W'\rangle$ is an isomorphism. Since $\ell_{\langle E\rangle}|W = \psi|W$ is an isomorphism onto W' and $P \notin \langle W_2\rangle$, we get $\ell_{\langle E\rangle}(P) \notin \langle W_1\rangle$. Since this is true for all $W_1 \subsetneq W$, Lemma 3 gives that W' evinces the border X_E-rank of the point $\ell_{\langle E\rangle}(P)$. Our choice of E implies $\mathcal{O}_{X_E}(2W') \neq \mathcal{O}_{X_E}(1)$. Hence part (b) of Proposition 2 gives the existence of a unique scheme $A \subset X_E$ such that $A \neq W'$ and $\ell_{\langle E\rangle}(P) \in \langle A\rangle$. Set $A_E := \psi^{-1}(A)$. Since $E \cap W = \emptyset$ and $\deg(A_E) = \deg(W)$, to prove $E + A_E \notin \mathbb{B}$ it is sufficient to prove $A_E \neq W$, i.e. (since ψ is an isomorphism) $W' \neq A$. We chosed $A \neq W$. Call $X[n-2w-1]$ the set of all E for which $E + A_E$ is defined.

(b) Let $\Gamma \subseteq \mathcal{S}$ be any irreducible component of $\mathcal{S}$ containing the irreducible algebraic family $\{E + A_E\}_{E\in X[n-2w-1]}$ constructed in step (a). Let F be a general element of Γ. Remember that to prove $r_X(P) = n + 1 - w$ it is sufficient to find a reduced $S \in \Gamma$. Γ is an irreducible algebraic family of divisors on X. We have $\dim(\Gamma) = n - 2w - 1$. By construction for a general $E \subset X$ such that $\sharp(E) = n-2w-1$ there is $B_E \in \Gamma$ such that $E \subset B_E$. For general E we have $\langle E\rangle \cap \langle W\rangle = \emptyset$. Since $P \notin \langle E\rangle$, the scheme $\ell_{\langle E\rangle}(W)$ is isomorphic to W, $P \in \langle \ell_{\langle E\rangle}(W)\rangle$ and $P \notin \langle W'\rangle$ for any $W' \subsetneq \ell_{\langle E\rangle}(W)$. Lemma 2 gives $\ell_{\langle E\rangle}(P) \notin \sigma_k(X_E)$ for general E. For general E the degree $2k + 2$ line bundles $\mathcal{O}_X(2W)$ and $\mathcal{O}_X(1)(-E)$ are not isomorphic. Thus part (b) of Proposition 2 applied to the curve X_E, the point $\ell_{\langle E\rangle}(P)$ and the scheme $Z := \ell_{\langle E\rangle}(W)$ gives that such a divisor B_E is unique. Thus Γ is an involution in the classical terminology ([6], §5). Assume for the moment that Γ has no fixed component. We get that either F is reduced (and hence parts (i) and (ii) of Theorem 1 are proved for P) or there is an integer $m \geq 2$ such that each connected component of F appears with multiplicity m ([6], Proposition 5.8). Since $F = E + A_E$ with E reduced and $\sharp(E) > \deg(A_E)$ this is absurd. Hence we may assume that Γ has a base locus. Call D the base locus of Γ. Thus the irreducible algebraic family $\Gamma(-D)$ of effective divisors of X has the same dimension and it is base point free. We have $F = D + F'$ with F' general in $\Gamma(-D)$. Since $\Gamma(-D)$ is an involution without base points and whose general member has at least one reduced connected component (a connected component of E), its general member F' is reduced ([6], Proposition 5.8). Since D has finite support and F' is general, we also have $F' \cap D = \emptyset$. Fix $O \in D_{red}$. We have $O \notin \langle W\rangle$, because $\deg(W \cup \{O\}) = w + 1$ and every degree $w + 1$ subscheme of X is linearly independent. Let E_1 be the union of O and $n - 2w - 2$ general point of X (if $m = 2w + 2$, then $E_1 = \{O\}$). Since $O \notin \langle W\rangle$ and X is non-degenerate, we have $\langle W\rangle \cap \langle E_1\rangle = \emptyset$. Thus the point $\ell_{\langle E_1\rangle}(P)$ is contained in the linear span of the degree w subscheme $\ell_{\langle E_1\rangle}(W)$ of the linearly normal elliptic curve $X_{E_1} \subset \mathbb{P}^{2w+2}$, but not in the linear span of any proper subscheme of it. Since any degree $2w + 1$ subscheme of X_{E_1} is linearly independent, we get $b_{X_{E_1}}(\ell_{\langle E_1\rangle}(P)) = w + 1$. Since O is a base point of Γ, we also get a one-dimensional family Γ' of distinct degree $w + 1$ subschemes of X_{E_1} such that $\ell_{\langle E_1\rangle}(P)$ is in the linear span of each of it. Part (a) of Proposition 2 gives that these schemes are pairwise disjoint. Hence $\deg(D) = 1$ and $D = \{O\}$ (as schemes). Since $E + A_E$ has at least $\deg(E_1)$ points with multiplicity one, at least one connected component of the general element F' of Γ' is reduced. Since F' is a general element of the base point free involution $\Gamma(-D)$, F' is reduced ([6], Proposition 5.8). Since any degree n divisor of X is linearly independent, we have

$\langle E_1\rangle \cap X = E_1$ (scheme-theoretic intersection). Since Γ' has no base points, we may also assume that $F' \cap (X_{E_1} \setminus \ell_{\langle E_1\rangle}(X \setminus E_1)) = \emptyset$. Hence the counterimage F'' of F' in X is disjoint from E_1. Thus $F'' \cup E_1$ is reduced. Since $P \in \langle F'' \cup E_1\rangle$, we get $r_X(P) \le n+1-w$.

A side remark. In the case $n \ge 2w-1$ we may even prove $D = \emptyset$. Indeed, assume $D \ne \emptyset$ and fix $O \in D_{red}$. Since $n-2w \ge 1$, in the previous construction we have $E_1 \ne \emptyset$. Since we may choose E_1 general after fixing both W and O, we get $\mathcal{O}_X(2W+O+E_1) \ne \mathcal{O}_X(1)$, contradicting part (b) of Proposition 2. □

Proposition 3. *Fix an integer $k \ge 1$ and a linearly normal elliptic curve $X \subset \mathbb{P}^{2k+1}$. Then there are $Q, P \in \mathbb{P}^{2k+1}$ such that $b_X(Q) = b_X(P) = r_X(Q) = k+1$ and $r_X(P) \ge k+2$. The set of all such points Q contains a non-empty open subset of $\mathbb{P}^{2k+1}$, while the set of all such points P contains a non-empty algebraic subset of codimension 2 of $\mathbb{P}^{2k+1}$.*

Proof. Since $\sigma_{k+1}(X) = \mathbb{P}^{2k+1}$, while $\dim(\sigma_k(X)) = 2k-1$ ([1], Remark 1.6), we may take as Q a general point of $\mathbb{P}^{2k+1}$. Now we prove the existence of points $P \in \mathbb{P}^n$ such that $r_X(P) > b_X(P) = k+1$ and that the set of all P such that $b_X(P) = k+1 < r_X(P)$ contains a codimension 2 subset of $\mathbb{P}^{2k+1}$. Let $\mathcal{U}$ be the set of all degree $k+1$ schemes $Z_1 \subset X$ such that Z_1 is unreduced and $2Z_1 \notin |\mathcal{O}_X(1)|$. The set $\mathcal{U}$ is a quasi-projective integral variety of dimension $k+1$. Fix any $Z_1 \in \mathcal{U}$. Let $\mathcal{V}(Z_1)$ denote the set of all unreduced $Z_2 \in |\mathcal{O}_X(1)(-Z_1)|$ such that $Z_2 \cap Z_1 = \emptyset$. The set $\mathcal{V}(Z_1)$ is a quasi-projective and integral variety of dimension k. Since $Z_1 \cap Z_2 = \emptyset$, Remark 1 shows that $\langle Z_1\rangle \cap \langle Z_2\rangle$ is a single point, Q. If $b_X(Q) = k+1$, then $\mathcal{Z}(X,Q) = \{Z_1, Z_2\}$, because $\mathcal{O}_X(2Z_1) \ne \mathcal{O}_X(1)$ (Part (b) of Proposition 2). Since neither Z_1 nor Z_2 is reduced, we get $r_X(Q) > k+1$. Varying Z_2 for a fixed Z_1 the set of all points Q obtained in this way covers a non-empty open subset of an irreducible hypersurface of $\langle Z_1\rangle$. Assume $b_X(Q) \le k$. and fix $W \in \mathcal{Z}(X,Q)$. Notice that $P \notin \langle W'\rangle$ for any $W' \subsetneqq W$. Since $\deg(W) + \deg(Z_1) \le n$, Lemma 1 and Remark 1 give the existence of $Z' \subsetneqq Z$ such that $Q \in \langle Z'\rangle$. Iterating the trick taking Z' and W instead of Z_1 and W we get $W \subseteqq Z'$ and hence $W \subset Z_1$ Making this construction using Z_2 and W we get $W \subsetneqq Z_2$. Since $Z_1 \cap Z_2 = \emptyset$, we obtained a contradiction. □

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